A QUESTION ABOUT Pic(X) AS A G-MODULE

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ABSTRACT. Let G be a finite group acting faithfully on an irreducible non-singular projective curve defined over an algebraically closed field F. Does every G-invariant divisor class contain a G-invariant divisor? The answer depends only on G and not on the curve. We answer the same question for degree 0 divisor (classes). We investigate the question for cycles on varieties.

CONTENTS

1. INTRODUCTION

This paper addresses the following classical question: Let X be an irreducible non-singular projective variety of dimension n defined over an algebraically closed field F and let G be a finite subgroup of the geometric automorphism group of X. That is, G is a finite group of automorphisms of the function field $F(X)$ that fixes F. Let D be an r-cycle and assume that its equivalence class $[D]$ is G-equivariant. Is there always a $D' \in [D]$ which is G-equivariant? This was addressed by Lonsted [Lo], who answered the question in the cyclic case for rational equivalence but left the general case open. We settle the case

of arbitrary finite groups in the case of curves and generalize his result for varieties. Let $Z^r(X)$ denote the group of $(n-r)$-cycles, let $\text{Rat}^r(X)$ denote the subgroup of those rationally equivalent to 0, let $\text{Alg}^r(X)$ denote the subgroup of those algebraically equivalent to 0, let $\text{Hom}^r(X)$ denote the subgroup of those homologically equivalent to 0, and let $\text{Num}^r(X)$ denote the subgroup of those numerically equivalent to 0. Let $D \in Z^r(X)$ be an $(n-r)$-cycle and assume that its class $[D]$ (with respect to one of these equivalences) is G-equivariant.

Question 1.1. Is there always a $D' \in [D]$ which is G-equivariant?

Section 1 addresses curves. Section 2 modifies the argument in the curve case, when possible, to varieties. For varieties, there is an analogous question for algebraic equivalence classes of divisors, and also for numerical equivalence classes of divisors. One surprising result is that, for K3 surfaces, if the Schur multiplier is trivial then the answer is yes, no matter which notion of equivalence one uses.

2. Cohomology

The basic idea is to use group cohomology to attack this question. For background on cohomology, we reference Serre [S], ch. VII. Though this is Lonsted's method as well, this paper was almost completely written before this was known. Consider the short exact sequences

$$1 \to F^\times \to F(X)^\times \to \text{Prin}(X) \to 0,$$

and

$$0 \to \text{Prin}(X) \to \text{Div}(X) \to \text{Pic}(X) \to 0,$$

where the additive group $\text{Prin}(X)$ of principal divisors may be identified with the multiplicative group $F(X)^\times/F^\times$ via the divisor map. Each of these is a $\mathbb{Z}[G]$-module. The covariant functor of G-invariants, $M \longmapsto H^0(G,M) = M^G$ is left exact. Therefore we have

$$1 \to H^0(G, F^\times) \to H^0(G, F(X)^\times) \to H^0(G, \text{Prin}(X))$$
$$(1) \qquad \to H^1(G, F^\times) \to H^1(G, F(X)^\times) \to H^1(G, \text{Prin}(X)) \to$$
$$H^2(G, F^\times) \to H^2(G, F(X)^\times) \to \dots,$$

and

$$(2) \qquad 0 \to H^0(G, \text{Prin}(X)) \to H^0(G, \text{Div}(X)) \to H^0(G, \text{Pic}(X))$$
$$\to H^1(G, \text{Prin}(X)) \to H^1(G, \text{Div}(X)) \to H^1(G, \text{Pic}(X)) \to \dots.$$

3. Curves

Let X be a curve. We claim that the answer to the above question is "no" in general.

Remark 3.1. The answer is "yes" in the special case when $[K]$ is the canonical class. The canonical class is $[df]$, where f is any function on the curve such that $df \neq 0$. Clearly, there exists such an f that is G-invariant.

Remark 3.2. There is a counterexample to the analogous question for number fields.

Proposition 3.3. $H^1(G, \mathrm{Div}(X)) = 0$.

Proof. For each $P \in X$, let $G_P = \{g \in G \mid gP = P\}$ denote its stabilizer (or inertia group). Set $L = L_P = \oplus_{g \in G/G_P} \mathbb{Z}[gP]$. As an abelian group, $\mathrm{Div}(X)$ can be decomposed into a direct sum of subgroups L_P, with one representative P from each orbit for the action of G on X. Note $L \cong \mathrm{Ind}_H^G(\mathbb{Z})$, where $H \cong G_P$ is the stabilizer of one of the basis elements. Here G acts on the induced module

$$\mathrm{Ind}_H^G(\mathbb{Z}) = \{f : G \to \mathbb{Z} \mid f(hg) = hf(g), \quad \forall h \in H, \ g \in G\},$$

which are just the $\mathbb{Z}$-valued functions on $H \backslash G$, by right multiplication. By Shapiro's Lemma, $H^1(G, \mathrm{Ind}_H^G(\mathbb{Z})) \cong H^1(H, \mathbb{Z})$. Now $H^1(H, Z) = \mathrm{Hom}(H, \mathbb{Z}) = 0$, since H is finite. $\square$

We want to know if $H^1(G, \mathrm{Prin}(X))$ is trivial or not, as then the answer to the question will follow from (2). Observe that, due to (1), $H^1(G, \mathrm{Prin}(X)) = 0$ if and only if the map $H^2(G, F^\times) \to H^2(G, F(X)^\times)$ is injective. The map $H^2(G, F^\times) \to H^2(G, F(X)^\times)$ is defined as follows: take the cocycle defining the extension of G by $F^\times$ associated to an element α of $H^2(G, F^\times)$, use it to define an extension of G by $F(X)^\times$ in the obvious way, then let the image of α be the class associated to this extension. More precisely: The group $H^2(G, A)$ classifies extensions E of the form

$$1 \to A \to E \to G \to 1.$$

We think of such a group E as a set of pairs (g, a) with group multiplication $(g, a)(g'a') = (gg', \beta(g, g')aa')$, for $g, g' \in G$, $a, a' \in A$ and the cocycle $\beta : G \times G \to A$ represents the associated class in $H^2(G, A)$. So, any extension E of G by $F^\times$ is associated to a cocycle $\beta : G \times G \to F^\times$. This may be "extended" (apologies for the over-use of this word) to an extension E' of G by $F(X)^\times$ associated to the *same* cocycle. The map $E \longmapsto E'$ defines the map $H^2(G, F^\times) \to H^2(G, F(X)^\times)$ in the above

long exact sequence. In particular, the answer to our question is yes if and only if each non-split (central) extension of G by $F^\times$ remains non-split when "extended" to $F(X)^\times$.

Now $H^2(G, F(X)^\times)) = 1$ by Tsen's theorem (a function field over an algebraically closed field is a C^1 field). This follows from the Corollaries on pages 96 and 109 of Shatz [Sh]. See also §4 and §7 of chapter X in [S]. By Tsen's theorem and (1), $H^1(G, \mathrm{Prin}(X)) \to H^2(G, F^\times)$ is surjective. To see that the map is also injective, note that $H^1(G, F(X)^\times) = 1$, by Hilbert's Theorem 90. So we have noted that:

Lemma 3.4. $H^1(G, \mathrm{Prin}(X)) \cong H^2(G, F^\times)$.

When $F = \mathbb{C}$, these two lemmas imply that there is a G-equivariant representative of every G-equivariant divisor class if and only if the Schur multiplier of G is trivial.

Theorem 3.5. *The sequence*

$$\mathrm{Div}(X)^G \to \mathrm{Pic}(X)^G \to H^2(G, F^\times) \to 0$$

is exact. In particular, there always a $D' \in [D]$ which is G-equivariant if and only if $H^2(G, F^\times) = 1$.

So there are many examples where the map on fixed points fails to be surjective.

Remark 3.6. Let p be the characteristic of F. If $p = 0$ then $H^2(G, F^\times)$ is the Schur multiplier of G. If $p > 0$ then $H^2(G, F^\times)$ is the p'-part of the Schur multiplier of G.

We give an easy corollary that has no cohomology in the statement.

Corollary 3.7. *Let G be a finite group.*
  (1) *Let G act on the curve X over the complex numbers. Suppose that $\mathrm{Div}(X)^G$ surjects onto $\mathrm{Pic}(X)^G$. Then $\mathrm{Div}(Y)^G$ surjects onto $\mathrm{Pic}(Y)^G$ for any curve Y over any algebraically closed field.*
  (2) *Let G act on the curve X_i over the algebraically closed field K_i for $i = 1, 2$. Suppose that $\mathrm{Div}(X_i)^G$ surjects onto $\mathrm{Pic}(X_i)^G$ for $i = 1, 2$, and suppose that the characteristics of K_1 and K_2 are distinct. Then $\mathrm{Div}(Y)^G$ surjects onto $\mathrm{Pic}(Y)^G$ surjects onto $\mathrm{Pic}(Y)^G$ for any curve Y over any algebraically closed field.*

If the Sylow ℓ-subgroup of G is cyclic, then the ℓ part of $H^2(G, F^\times)$ is trivial (since the restriction map to the Sylow ℓ-subgroup is an injection on the ℓ part of cohomology), if all Sylow subgroups are cyclic, then $H^2(G, F^\times)$ is trivial. Such groups are well known to be metacyclic. This yields:

Corollary 3.8. *If every ℓ-Sylow subgroup of G is cyclic (for every prime ℓ dividing $|G|$), then $\mathrm{Pic}(X)^G/\mathrm{Div}(X)^G = 0$. In particular, if G is cyclic then for each G-invariant divisor class $[D]$ there is always a $D' \in [D]$ which is G-equivariant.*

We next want to consider $\mathrm{Pic}_0(X)$ (i.e. the Jacobian) and show that the map $\phi : \mathrm{Div}_0(X)^G \to \mathrm{Pic}_0(X)^G$ may also fail to be surjective. Consider the degree map deg on $\mathrm{Div}(X)$ and also on $\mathrm{Pic}(X)$. Let B be $\deg(\mathrm{Div}(X)^G)$. We identify B with a subgroup of $\mathrm{Pic}(X)/\mathrm{Pic}_0(X) = \mathbb{Z}$. Clearly, B contains $|G|\mathbb{Z}$ and may be bigger. There is no analog of Proposition 3.3. Instead, we have the following result.

Lemma 3.9. *Let $b = \gcd\{|G|/|I|\}$ as $I \subseteq G$ ranges over the inertia subgroups of G, and set $B = b\mathbb{Z}$. Then $H^1(G, \mathrm{Div}_0(X)) \cong \mathbb{Z}/B$*

Proof. Consider the sequence $0 \to \mathrm{Div}_0(X) \to \mathrm{Div}(X) \to \mathbb{Z} \to 0$. The map from $\mathrm{Div}(X)$ to $\mathbb{Z}$ is deg. Using Proposition 3.3 yields:

$$(0 \to \mathrm{Div}_0(X)^G \to)\mathrm{Div}(X)^G \to \mathbb{Z} \to H^1(G, \mathrm{Div}_0(X)) \to 0,$$

as asserted. Now we prove the claim about b. Observe that a generating set of $\mathrm{Div}(X)^G$ are the sum of points in a single G-orbit. If I is the inertia group a point in the orbit, then the degree of this divisor is $[G : I]$. Thus the image of $\deg(\mathrm{Div}(X)^G) = B$, where B is described in the theorem. $\square$

Now consider the short exact sequence

$$0 \to \mathrm{Prin}(X) \to \mathrm{Div}_0(X) \to \mathrm{Pic}_0(X) \to 0.$$

Taking fixed points leads to the long exact sequence for cohomology:

$$0 \to \mathrm{Prin}(X)^G \to \mathrm{Div}_0(X)^G \overset{\phi}{\to} \mathrm{Pic}_0(X)^G \to$$
$$H^1(G, \mathrm{Prin}(X)) \to H^1(G, \mathrm{Div}_0(X)) \to H^1(G, \mathrm{Pic}_0(X)).$$

This and Lemma 3.4 proves the following result.

Theorem 3.10. *There is an isomorphism*

$$\mathrm{Pic}_0(X)^G/\phi(\mathrm{Div}_0(X)^G) \cong \ker\{H^1(G, \mathrm{Prin}(X)) \to H^1(G, \mathrm{Div}_0(X))\}.$$

In particular, if $H^2(G, F^\times) = 1$ then for each G-invariant divisor class $[D]$ of degree 0 there is always a degree 0 divisor $D' \in [D]$ which is G-equivariant.

Next we want to identify the image of $H^1(G, \mathrm{Prin}(X))$ in $\mathbb{Z}/B$, where B is defined above. Now deg induces a map from $\mathrm{Pic}(X)^G$ to $\mathbb{Z}$. Denote this image by A (and note that it contains B the image of

$\mathrm{Div}(X)^G$). Indeed, since $\mathrm{Pic}_0(X)^G$ is the kernel of deg, this shows that $\mathrm{Pic}(X)^G/(\mathrm{Pic}_0(X)^G I) \cong A/B$. Thus,

$$0 \to \mathrm{Pic}_0(X)^G/I_0 \to \mathrm{Pic}(X)^G/I \to A/B \to 0.$$

This implies that the kernel of the map going from $H^1(G, \mathrm{Prin}(X))$ to $H^1(G, \mathrm{Div}_0(X))$ has order $|A/B|$ and since the image is cyclic, this yields:

Proposition 3.11. *The image of $H^1(G, \mathrm{Prin}(X))$ in $H^1(G, \mathrm{Div}_0(X))$ is isomorphic to A/B where $A = \deg(\mathrm{Pic}(X)^G)$ and $B = \deg(\mathrm{Div}(X)^G)$.*

Now $H^1(G, \mathrm{Prin}(X)) \cong H^2(G, F^\times)$ can be essentially any abelian group one wishes (by taking direct products for example). On the other hand, we saw that $H^1(G, \mathrm{Div}_0(X))$ is a finite cyclic group. So there are certainly examples where this kernel is nontrivial (and as large as one wishes). We now give some examples:

Example 3.12. We work over the field of complex numbers. If $X = \mathbb{P}^1$ then $\mathrm{Pic}(X) = \mathbb{Z}$ and $H^0(G, \mathrm{Pic}(X)) = \mathbb{Z}$ as well. Since $\mathrm{Pic}(X)_0 = 0$, the answer to the question is "yes" if and only if there is a fixed divisor of any given degree (since there is only one class with a given degree). We show that this can fail. Fix an embedding of $G = A_5$ into $PGL(2, \mathbb{C})$ (to get an action of G on $\mathbb{P}^1$). In this case, inertia groups (i.e. stabilizers of points on the curve) are always cyclic, so the only possibilities are of order 1, 2, 3 and 5. Thus the possible orbit sizes are 12, 20, 30 and 60 and any G-fixed divisor has degree a multiple of the greatest common divisor of these numbers (i.e. 2). Now Theorem 3.5 implies that $H^2(A_5, \mathbb{C}^\times) = \mathrm{Pic}(X)^G/\mathrm{Div}(X)^G = \mathbb{Z}/2\mathbb{Z}$, which of course is well-known (see e.g. [K], pp. 245–246).

Now let F be an algebraically closed field of characteristic 2. Then $H^2(A_5, F^\times) = 0$, since the $2'$ part of the Schur multiplier of A_5 is zero. Fix an embedding of A_5 into $PGL(2, F)$ so that A_5 acts on $X = \mathbb{P}^1$. It follows from Theorem 3.5 that $\mathrm{Div}(X)^G = \mathrm{Pic}(X)^G = \mathbb{Z}$. Indeed, the inertia groups are A_4 and cyclic of order 5, and the gcd of their indices is one.

Example 3.13. Let F be any closed field of characteristic not 2. Let G be an elementary abelian group of order 4 acting on the curve $\mathbb{P}^1(F)$. One can argue as in the previous example that $H^2(G, F^\times) = \mathbb{Z}/2\mathbb{Z}$.

Example 3.14. Let F be any closed field of characteristic not 2. Let G be an elementary abelian group of order 2^r acting on the curve X. If G acts fixed point freely on X, then $B = 2^r \mathbb{Z}$ in the notation above. Since A/B is cyclic and embeds in $H^2(G, F^\times)$, a group of exponent 2, it follows that every G-invariant divisor class on X has degree a multiple

of 2^{r-1}. We show below that there may be such a class or there may not be. Suppose that G does not act fixed point freely on X. Since G acts tamely on X, all inertia groups are cyclic and so have index a multiple of 2^{r-1}, it follows that $B = 2^{r-1}\mathbb{Z}$ and so every G-invariant divisor class has degree a multiple of 2^{r-2}.

We have already observed that there are exact sequences:

(3) $\mathrm{Div}^G(X) \to \mathrm{Pic}^G(X) \to H^2(G, F^\times) \to 0$ and

(4) $\mathrm{Div}_0^G(X) \to \mathrm{Pic}_0^G(X) \to H^2(G, F^\times) \to A/B \to 0,$

where $A = \deg(\mathrm{Pic}(X)^G)$ and $B = \deg(\mathrm{Div}(X)^G)$. Note that B is generated by the gcd of $|G : \mathrm{Stab}_G(x)|$ as x ranges over X.

It is convenient to define $C = C(X, G) = A/B$. We summarize some properties of this group.

Lemma 3.15. *We have:*

  (1) *The group $C = A/B$ is finite and cyclic.*
  (2) *$|C|$ divides the gcd of the $|G|/|I|$ as I ranges over the inertia groups.*
  (3) *$C = 1$ if there is a totally ramified point in X.*

Only the third statement deserves comment. Indeed, if the point $P \in X$ is totally ramified, then $[P]$ is a G-invariant degree one divisor.

David Saltman has shown us how to explicitly compute the last map $H^2(G, F^\times) \to A/B$ in the second statement. He has kindly allowed us to include his observations here. Let n be an integer coprime to the characteristic of F. Let c be a 2-cocycle in $H^2(G, \mu_n)$. Then $c = \delta(d_g)$, where d_g is a coboundary with values in $F(X)^\times$ by Tsen's theorem. Since d_g^n is trivial in H^1, it follows that

(5) $d_g^n = g\theta/\theta$

for some function θ on X. Now $F(X)(\theta^{1/n})/F(X)^G$ is a Galois cover that realizes c. The class of d_g vanishes in $H^1(G, \mathrm{Div}(X))$ and so we can write (switching now from multiplicative to additive notation)

$$(d_g) = gD - D$$

for a divisor D. Now multiply by n and use (5) to get $(g\theta) - (\theta) = ngD - D$. Hence

$$(\theta) - D \quad \text{is a G-invariant divisor.}$$

Now D is not well-defined – but its degree is well-defined modulo b.

The map in question $H^2(G, F^\times) \to A/B$ sends (the class of) c to the degree of D via the construction above.

Note that the question of how many G-invariant divisor classes are not in the image of a G-invariant divisor depends only on the group. We show that this is not the case for the analogous question for degree 0 divisors and divisor classes. The answer depends on the curve even if we assume that $X \to X/G$ is unramified.

We have already set $C(X, G)$ to be $\deg(\mathrm{Div}(X)^G)/\deg(\mathrm{Pic}(X)^G)$. By definition, $C(X, G)$ equals A/B. We already observed that the term $H^2(G, F^\times)$ in (3) does not depend on the curve X. So it seems natural to ask whether the term $C = C(X, G)$ only depends on G and the ramification data of X. We next show Theorem 3.18 that indeed, this fails over the complex field even for G equals the Klein four group acting without fixed points.

We will need two lemmas to prove the theorem.

Lemma 3.16. *Let the finite group G act on the curve X. Let N be a normal subgroup of G that acts on X without fixed points. Let Q be the quotient group and let Y be the quotient curve. Then $|C(Y, Q)|$ divides $|C(X, G)|$.*

Proof. A Q-invariant divisor class on Y of degree d lifts to a G-invariant divisor class of degree $|N|d$ on X. However, for divisors, you can also go down: a G-invariant divisor of degree $|N|d$ on X always comes from a Q-invariant divisor on Y of degree d. $\square$

Lemma 3.17. *Let X be a curve over the algebraically closed field F of characteristic not 2. Let G be an elementary abelian 2-group of order 2^r with $r \geq 3$. Then the cokernel of $\mathrm{Div}(X)^G \to \mathrm{Pic}(X)^G$ is an elementary abelian group of order 2^m where $m = r(r-1)/2$.*

Proof. The lemma follows from the fact that $H^2(G, F^\times)$ equals $\wedge^2(G)$ and from Theorem 3.5. $\square$

Theorem 3.18. *Let K be the Klein four group.*
  (1) *There exists a curve E over the field of complex numbers such that K acts without fixed points on E and $C(E, K) = \mathbb{Z}/2\mathbb{Z}$.*
  (2) *There exists a curve Y over the field of complex numbers such that K acts without fixed points on Y and $C(Y, K) = 1$.*

Proof. Let E be an elliptic curve over $\mathbb{C}$ and let K act by translations on Y. Note that indeed K acts without fixed points on E. In particular, any K-invariant divisor on E has degree 4.

Let $\{0, P, Q, R, \}$ be the of two-torsion of $E(\mathbb{C})$. Using the group law on E we have $0 + P = Q + R$, which implies that as divisors

$(0) + (P) \sim (Q) + (R)$. Note further $E(\mathbb{C})$ and in particular K acts trivially on $\mathrm{Pic}_0(E)$. Thus there is an invariant divisor class of degree 2. Thus $C(E, K) = \mathbb{Z}/2\mathbb{Z}$ and the first part is proved.

Let $r \geq 3$ be a positive integer. Let J be the affine group $AGL(r, 2)$. The subgroup of translations G is elementary abelian of order 2^r. Note that G is normal in J. Now J acts on some curve X with the cover $X \to X/J$ unramified (in characteristic 0, this follows from the description of the fundamental group – in positive characteristic, this follows from [Ste]). Now we note that it follows from our proofs above that the map from $H^1(G, \mathrm{Prin}(X)) \to H^1(G, \mathrm{Div}_0(X))$ is J-equivariant. Since $H^1(G, \mathrm{Prin}(X)) \cong \wedge^2(G)$ is J-irreducible and not cyclic (since $r \geq 3$), it follows that $C(X, G)$ is trivial. Now let N be a subgroup of G of index 4. We have $K = G/N$ and set $Z = X/N$. Now lemma 3.16 implies that $C(Z, K)$ is trivial. So we have constructed two curves with the same group acting without fixed points such that the cokernels of $\mathrm{Div}_0(X)^G \to \mathrm{Pic}_0(X)^G$ are different. The theorem is proved. $\square$

One can choose the two curves to have the same genus as well.

4. Varieties

In this section, X is a non-singular, projective variety over the field F of complex numbers. Let D and D' be $(n-r)$-cycles on X. Recall that if D and D' are rationally equivalent then they are algebraically equivalent, if D and D' are algebraically equivalent then they are homologically equivalent, if D and D' are homologically equivalent then they are numerically equivalent ([Har], §V.1, Exer. 1.7, and [Ful1], §19.3). Using the notation from the introduction above, for $D \in \mathrm{Div}(X)$,

$$[D]_r = D + Rat^r(X) \subset [D]_a = D + \mathrm{Alg}^r(X) \subset$$
$$[D]_h = D + \mathrm{Hom}^r(X) \subset [D]_n = D + \mathrm{Num}^r(X).$$

When specifying which notion of equivalence is not needed, we drop the subscript.

Lemma 4.1. *If $[D]_r$ is G-equivariant then so are $[D]_a$, $[D]_h$, and $[D]_n$.*

Proof. If $[D]$ is G-equivariant if and only if for all $g \in G$, $gD - D$ is equivalent to 0. Since $Rat^r(X) \subset \mathrm{Alg}^r(X) \subset \mathrm{Hom}^r(X) \subset \mathrm{Num}^r(X)$, we have the result claimed in the lemma. $\square$

Consider again the exact sequences (1) and (2), where X is now a variety.

4.1. **Divisors.** We start with the analog of a result in the previous section.

Lemma 4.2. *The map $H^1(G, \mathrm{Prin}(X)) \to H^2(G, F^\times)$ is injective.*

Proof. By Hilbert's Theorem 90, $H^1(G, F(X)^\times) = 1$, so this follows from (1). $\square$

As in the case of curves, it follows that there is a G-equivariant representative of a G-equivariant divisor class if the Schur multiplier of G is trivial.

Theorem 4.3. *If $H^2(G, F^\times) = 1$ then the map $\mathrm{Div}(X)^G \to \mathrm{Pic}(X)^G$ is surjective. In other words, if $H^2(G, F^\times) = 1$ and $[D]_r$ is G-invariant then there always a $D' \in [D]_r$ which is also G-equivariant.*

Corollary 4.4. *If every ℓ-Sylow subgroup of G is cyclic (for every prime ℓ dividing $|G|$) then for each G-invariant divisor class $[D]_r$ there is always a $D' \in [D]_r$ which is G-equivariant.*

Recall there is a natural 1-1 correspondence between invertible subsheafs $\mathcal{L}(D)$ of the sheaf $\mathcal{K}$ of total quotient rings on X and divisors D ([Har], §II.6).

Corollary 4.5. *If the hypothesis to Corollary 4.4 holds then for each G-equivariant invertible subsheaf $\mathcal{L}$ of $\mathcal{K}$ on X there is a G-equivariant divisor D of X such that $\mathcal{L} = \mathcal{L}(D)$.*

Proof. Use the above correspondence and the previous corollary. $\square$

The long exact sequence (1) does not help to determine the image of the map
$$\mathrm{Div}(X)^G \to (\mathrm{Div}(X)/\mathrm{Alg}^1(X))^G,$$
or of the map
$$\mathrm{Div}(X)^G \to (\mathrm{Div}(X)/\mathrm{Num}^1(X))^G.$$
However, in some special cases, one can say more.

Theorem 4.6. *Let X be a $K3$ surface. If $H^2(G, F^\times) = 1$ then the map $\mathrm{Div}(X)^G \to (\mathrm{Div}(X)/\mathrm{Alg}^1(X))^G$, is surjective. In other words, if $H^2(G, F^\times) = 1$ and $[D]_a$ is G-invariant then there always a $D' \in [D]_a$ which is G-equivariant. The analogous statements with Alg^1 replaced by Hom^1 or Num^1 also hold.*

Proof. For $K3$ surfaces, $[D]_r = [D]_a = [D]_h = [D]_n$ ([SD], §2.3), so this is a consequence of the analogous result proven previously for rational equivalence. $\square$

4.2. **Toric varieties.** In this subsection, we show that the answer to the question asked in our introduction is "yes" for rational cycle classes on toric varieties, at least if G consists only of "toric automorphisms". In the toric case, the Chow groups can be described in terms of the combinatorial geometry of the fan associated to the variety. Let Δ be a finite, non-singular, strongly polytopal, complete fan associated to an integral lattice L in $\mathbb{R}^n$ and let $X = X(\Delta)$ denote the associated toric variety. (See Definition V.4.3 of [Ewa] for the definition of strongly polytopal.) The variety X is complete and projective and contains a torus T densely. A **toric automorphism** is a T-equivariant automorphism of $X(\Delta)$.

Theorem 4.7. *Suppose that*

(a) *$G \subset \mathrm{Aut}(X)$ is a finite subgroup of the group of toric automorphisms of X; and*

(b) *the class $[Z] \in Z^r(X)/Rat^r(X)$ is G-equivariant. Then there is a $Z' \in [Z]$ which is G-equivariant.*

Proof. Toric automorphisms of X correspond to automorphisms of the lattices which also preserve Δ (Theorem 1.13 in Oda [O]). We know by the Proposition in §5.1 in [Ful2] that $[Z]$ is an integral combination of the classes of the orbit closures $V(\sigma)$, σ a cone in Δ:

$$[Z]_r = \sum_i n_i [V(\sigma_i)]_r, \qquad n_i \in \mathbb{Z}.$$

But such a combination of classes is fixed by G only if only if the cones $\{\sigma_i\}$ decomposes into a disjoint union of orbits and the n_i's are constant on each of these primitive G-orbits. In this case, the cycle $\sum_i n_i V(\sigma_i)$ itself is G-equivariant. $\square$

Remark 4.8. Let X be a non-singular variety for which $H^2(X, \mathbb{Z})$ is torsion-free. (This is true for toric surfaces, for example [Ful2], §3.4.) Then algebraic, homological and numerical equivalence of divisors agree ([Ful1], §19.3.1). Consequently, if $Z^r(X)^G \to (Z^r(X)/\mathrm{Alg}^r(X))^G$ is surjective then so is $Z^r(X)^G \to (Z^r(X)/\mathrm{Num}^r(X))^G$, and conversely.

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